A JUZVINSKIĬ ADDITION THEOREM FOR FINITELY GENERATED FREE GROUPS ACTIONS

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Abstract. The classical *Juzvinskiĭ Addition Theorem* states that the entropy of an automorphism of a compact group decomposes along invariant subgroups. Thomas generalized the theorem to a skew-product setting. Using L. Bowen's *f-invariant* we prove the addition theorem for actions of finitely generated free groups on skew-products with compact totally disconnected groups or compact Lie groups (correcting an error from [Bo10c]) and discuss examples.

Contents

1. Introduction

The following result was proven independently by H. Li [Li11] and Lind-Schmidt [LS09].

Theorem 1.1. *[Addition theorem for amenable groups] Let Γ be a countable discrete amenable group, G be a compact metrizable group and $\alpha : \Gamma \to Aut(G)$ an action of Γ on G by group-automorphisms. Suppose $N \lhd G$ is a closed normal $\alpha(\Gamma)$-invariant subgroup. Denote by $\alpha_N : \Gamma \to Aut(N)$ and $\alpha_{G/N} : \Gamma \to Aut(G/N)$ the induced actions and by $\mu_G, \mu_N, \mu_{G/N}$*

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the Haar probability measures on G, N and G/N respectively. Then the entropies of these actions satisfy:

$$h_{\mu_G}(\alpha) = h_{\mu_N}(\alpha_N) + h_{\mu_{G/N}}(\alpha_{G/N}).$$

In the case $\Gamma = \mathbb{Z}$, this result is due to Juzvinskiĭ [Ju65] from which it receives its name. The case $\Gamma = \mathbb{Z}^d$ was proven in [LSW90]. Special cases were obtained by Miles [Mi08] and Björklund-Miles [BM09].

The paper [Bo10a] introduced a measure-conjugacy invariant, called the *f-invariant*, for probability-measure-preserving actions of finitely generated free groups. (Later a more general theory of sofic entropy was introduced in [Bo10b], of which we have little to say in the present article). In [Bo10c], a proof is claimed that the above addition formula extends to the case when Γ is a finitely generated free group, the entropy is replaced with the f-invariant, and G is either totally disconnected, a Lie group, or a connected abelian group (whenever the f-invariant is well-defined). However, there is an error in the proof. We prove here that the statement remains correct if either G is totally disconnected (and a mild additional hypothesis is satisfied) or G is a Lie group and the action is by smooth automorphisms. See the corrigendum [BG12] for other corrections to [Bo10c]. The main result is Theorem 2.2 below. We also prove a skew-product addition formula in Theorem 3.3 which may be of independent interest.

Organization. §2 reviews the f-invariant and states the main theorem; §3 reviews skew-products and proves Theorem 2.2 from Theorem 3.3. In §5 and §6 Theorem 3.3 is proven; §7 discusses examples, including the Ornstein-Weiss example.

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2. The f-invariant

Let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be a rank r free group. Let α be a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$. We consider α as a homomorphism from Γ to the group of automorphisms of $(X, \mathcal{B}_X, \nu)$ and write α_g for $\alpha(g)$ ($\forall g \in \Gamma$). Let $\mathcal{P} = \{P_1, P_2, \ldots\}$ be a countable partition of X into measurable subsets. The Shannon-entropy of $\mathcal{P}$ is

$$H_\nu(\mathcal{P}) \triangleq -\sum_{P \in \mathcal{P}} \nu(P) \log(\nu(P)).$$

By convention $0 \log(0) \triangleq 0$. If $\mathcal{P}, \mathcal{Q}$ are two partitions of X then their *join* is defined by $\mathcal{P} \vee \mathcal{Q} \triangleq \{P \cap Q : \ P \in \mathcal{P}, Q \in \mathcal{Q}\}$. If $W \subset \Gamma$ is finite, we let $\mathcal{P}^W \triangleq \bigvee_{w \in W} \alpha_w \mathcal{P}$. Note that α is only implicit in this notation. If $H_\nu(\mathcal{P}) < \infty$ then define

$$\begin{aligned} F_\nu(\alpha, \mathcal{P}) &= (1-2r) H_\nu(\mathcal{P}) + \sum_{i=1}^r H_\nu(\mathcal{P} \vee \alpha_{s_i} \mathcal{P}) \\ f_\nu(\alpha, \mathcal{P}) &= \inf_{n>0} F_\nu\left(\alpha, \mathcal{P}^{B(n)}\right) \end{aligned}$$

where $B(n)$ is the ball of radius n centered at the identity with respect to the word metric. The partition $\mathcal{P}$ is said to be *generating* (for the action α) if the smallest $\alpha(\Gamma)$-invariant σ-algebra containing $\mathcal{P}$ is $\mathcal{B}_X$ (up to sets of measure zero). In [Bo10a] it is shown that if $\mathcal{P}, \mathcal{Q}$ are finite-entropy generating partitions then $f_\nu(\alpha, \mathcal{P}) = f_\nu(\alpha, \mathcal{Q})$. So we define the f-invariant of the action by $f_\nu(\alpha) \triangleq f_\nu(\alpha, \mathcal{P})$ where $\mathcal{P}$ is any finite-entropy generating partition for α. If there does not exist a finite-entropy generating partition for α then $f_\nu(\alpha)$ is undefined.

It will be useful to have an alternative formulation of the f-invariant for which we need the following definitions. For $g \in \Gamma$, let $h_\nu(\alpha_g, \mathcal{P})$ denote the entropy rate of $\mathcal{P}$ with respect to the $\mathbb{Z}$-action generated by α_g. To be precise,

$$h_\nu(\alpha_g, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{2n+1} H\left(\bigvee_{i=-n}^{n} \alpha_g^i \mathcal{P}\right).$$

The *entropy* of the action α_g is $h_\nu(\alpha_g) = \sup_{\mathcal{P}} h_\nu(\alpha_g, \mathcal{P})$ where the supremum is over all finite measurable partitions $\mathcal{P}$ of X. Define

$$\begin{aligned} F^*_\nu(\alpha, \mathcal{P}) &= (1-r)H_\nu(\mathcal{P}) + \sum_{i=1}^{r} h_\nu(\alpha_{s_i}, \mathcal{P}) \\ f^*_\nu(\alpha, \mathcal{P}) &= \inf_{n>0} F^*_\nu\left(\alpha, \mathcal{P}^{B(n)}\right). \end{aligned}$$

One of the main results of [Bo10c] is:

Theorem 2.1. *Let α be a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$. Then for any finite-entropy generating partition $\mathcal{P}$ for α, $f_\nu(\alpha) = f^*_\nu(\alpha, \mathcal{P})$.*

The main result of this paper is:

Theorem 2.2. *Let $\Gamma = \langle s_1, \ldots, s_r\rangle$ be a rank r free group, G be a compact metrizable group and $\alpha : \Gamma \to Aut(G)$ an action of Γ on G by group-automorphisms. Suppose $N \lhd G$ is a closed normal $\alpha(\Gamma)$-invariant subgroup. Denote by $\alpha_N : \Gamma \to Aut(N)$ and $\alpha_{G/N} : \Gamma \to Aut(G/N)$ the induced actions and by $\mu_G, \mu_N, \mu_{G/N}$ the Haar probability measures on G, N and G/N respectively. Suppose there exists finite-entropy generating partitions for $\alpha, \alpha_N, \alpha_{G/N}$ and one of the following hold.*

1. *N is totally disconnected and there exists a clopen finite-index normal subgroup $N_0 \lhd N$ such that $\{gN_0 : \ g \in N\}$ is a generating partition for α_N.*
2. *G is a compact Lie group and the action α is by smooth automorphisms.*

Then

$$f_{\mu_G}(\alpha) = f_{\mu_N}(\alpha_N) + f_{\mu_{G/N}}(\alpha_{G/N}).$$

Remark 2.1. The proof shows slightly more: if case (1) occurs and $\alpha_{G/N}$ has a finite-entropy generating partition, then α automatically has a finite-entropy generating partition. This follows from Lemmas 3.2 and 6.4. To be more precise, Lemma 3.2 shows that α is measurably conjugate to a skew-product action of the form $\alpha_{G/N} \times_\sigma \alpha_N$. If $\mathcal{P}$ is a finite-entropy generating partition for $\alpha_{G/N}$ and $\mathcal{Q} = \{gN_0 : \ g \in G\}$ is a generating partition for α_N of the kind described in case (1) above, then Lemma 6.4 shows that $\mathcal{P} \times \mathcal{Q}$ is generating for $\alpha_{G/N} \times_\sigma \alpha_N$. Because $\mathcal{P}$ has finite-entropy and $\mathcal{Q}$ is finite, $\mathcal{P} \times \mathcal{Q}$ has finite entropy as required.

Remark 2.2. Suppose N as above is totally disconnected and $N_0 \lhd N$ is a closed finite-index normal subgroup (the fact N_0 is closed and finite-index implies N_0 is clopen). Let $M = \bigcap_{g\in\Gamma} \alpha_g N_0$. Note M is $\alpha(\Gamma)$-invariant. Let $\alpha_{G/M}, \alpha_{N/M}$ be the induced actions on G/M and N/M respectively. Let $\mu_{G/M}, \mu_{N/M}$ be the respective Haar probability measures. Suppose that $\alpha_{G/M}$ and $\alpha_{G/N}$ admit finite-entropy generating partitions. Note that the clopen partition $\{gN_0/M : \ g \in \ G\}$ is a finite generating partition for $\alpha_{N/M}$, as it is the image of the clopen generating partion $\{gN_0 : \ g \in \ G\}$ under the continuous projection $\pi : N \to N/M$. So the theorem above implies

$$f_{\mu_{G/M}}(\alpha_{G/M}) = f_{\mu_{N/M}}(\alpha_{N/M}) + f_{\mu_{G/N}}(\alpha_{G/N}).$$

By the previous remark, this formula holds as long as $\alpha_{G/N}$ admits a finite-entropy generating partition.

3. Skew-products

The proof of Theorem 2.2 is based on a more general skew-product theorem of independent interest, the construction of which we recall next.

Definition 3.1. Let Γ be a group. Let $(X, \mathcal{B}_X, \nu)$ be a Lebesgue space equipped with a Γ-action α and G be a compact group with Borel σ-algebra $\mathscr{B}$ and Haar measure μ. Let β be a Γ-action by group-automorphisms on G. Let $\sigma : \Gamma \times X \to \mathrm{G}$ be a cocycle for β and α, i.e., σ is a measurable mapping so that for all $g, h \in \Gamma$, $x \in X$

$$\sigma(gh, x) = (\beta_g \sigma(h, x)) \cdot \sigma(g, \alpha_h x). \tag{3.1}$$

Define the *skew-product action* $\alpha \times_\sigma \beta$ *of* Γ *on* $X \times \mathrm{G}$ *by:*

$$(\alpha \times_\sigma \beta)_g(x, y) = (\alpha_g x, (\beta_g y) \cdot \sigma(g, x)) \ (g \in \Gamma, \, x \in X, y \in \mathrm{G}).$$

The connection between skew-product actions and the addition theorem is the following standard result (which we obtained from [Li11, Proof of Corollary 6.3]).

Lemma 3.2. *Let Γ be a countable group, G be a compact metrizable group, $\alpha : \Gamma \to Aut(G)$ an action of Γ on G by group-automorphisms and $N \lhd G$ a closed normal $\alpha(\Gamma)$-invariant subgroup. Denote by $\alpha_N : \Gamma \to Aut(N)$ and $\alpha_{G/N} : \Gamma \to Aut(G/N)$ the induced actions. Then there is a cocycle $\sigma : \Gamma \times G/N \to N$ such that $\alpha_{G/N} \times_\sigma \alpha_N$ is measurably conjugate with α.*

The main technical result of this paper is:

Theorem 3.3. *Let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be a rank r free group, α a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$, G a compact metrizable group, β an action of Γ on G by group-automorphisms, and $\sigma : \Gamma \times X \to G$ a cocycle for these actions. Suppose that G is totally disconnected and there exists a finite-index clopen normal subgroup $N \lhd G$ such that $\{gN : \ g \in G\}$ is a generating partition for β. Let μ denote the Haar probability measure on G. Suppose also that there is a finite-entropy generating partition for α. Then*

$$f_{\nu\times\mu}(\alpha \times_\sigma \beta) = f_\nu(\alpha) + f_\mu(\beta).$$

The analog of this theorem for discrete countable amenable groups Γ when G is an arbitrary compact metrizable group was established in [Li11]. The case $\Gamma = \mathbb{Z}$ was proven earlier by Thomas [Th71] and the case $\Gamma = \mathbb{Z}^d$ is shown in [LSW90].

Theorem 3.3 is proven in the next section. Next we combine this result with the following two lemmas to complete the proof of Theorem 2.2.

Lemma 3.4. *Let M be a smooth compact Riemannian manifold. Let $T : M \to M$ be a diffeomorphism. Then $h_\mu(T) < \infty$ for any T-invariant probability measure μ.*

Proof. This is due to Kushnirenko [Ku65]. Alternatively, it follows from Ruelle's inequality (see e.g. [KH95, Corollary S.2.17]). □

Lemma 3.5. *Let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be a rank r free group with $r > 1$, M be a smooth compact Riemannian manifold, α a measure-preserving action of Γ on M by diffeomorphisms and μ a non-atomic $\alpha(\Gamma)$-invariant probability measure on M. Then $f_\mu(\alpha) = -\infty$ if there is a finite-entropy generating partition for the action.*

Proof. Let $m = \max_{i=1}^r h_\mu(\alpha_{s_i})$. By the previous lemma, $m < \infty$. Let $\mathcal{P}$ be a finite-entropy generating partition for α. Let $N > 0$. Because μ is non-atomic, there is a finite partition $\mathcal{Q}$ of M with $H_\mu(\mathcal{Q}) > N$. So after replacing $\mathcal{P}$ with $\mathcal{P} \vee \mathcal{Q}$ if necessary, we may assume that $H_\mu(\mathcal{P}) > N$. By Theorem 2.1

$$\begin{aligned} f_\mu(\alpha) &= f^*_\mu(\alpha, \mathcal{P}) = \inf_{n>0} F^*_\mu(\alpha, \mathcal{P}^{B(n)}) \\ &\leq (1-r)H_\mu(\mathcal{P}) + \sum_{i=1}^r h_\mu(\alpha_{s_i}, \mathcal{P}) \\ &\leq (1-r)N + rm. \end{aligned}$$

Because $N > 0$ is arbitrary and $r > 1$, this implies the lemma. □

Proof of Theorem 2.2 from Theorem 3.3. Suppose item (1) holds. By Lemma 3.2, α is measurable conjugate with $\alpha_{G/N} \times_\sigma \alpha_N$ for some cocycle σ. So Theorem 3.3 implies

$$f_{\mu_G}(\alpha) = f_{\mu_G}(\alpha_{G/N} \times_\sigma \alpha_N) = f_{\mu_{G/N}}(\alpha_{G/N}) + f_{\mu_N}(\alpha_N)$$

as required.

Suppose that item (2) holds; i.e., G is a compact Lie group and α is an action by smooth group-automorphisms. If G is finite then the theorem is clear because

$$\begin{aligned} f_{\mu_G}(\alpha) &= -(r-1)\log|G| = -(r-1)\log|G/N| - (r-1)\log|N| \\ &= f_{\mu_{G/N}}(\alpha_{G/N}) + f_{\mu_N}(\alpha_N). \end{aligned}$$

By Theorem 1.1, we may assume, without loss of generality, that $r > 1$. If G is infinite then, because it is compact, it has positive dimension. So μ_G is non-atomic and the previous lemma implies $f_{\mu_G}(\alpha) = -\infty$. Also, because G is infinite, either N or G/N is infinite. Therefore, either μ_N or $\mu_{G/N}$ is non-atomic. Of course, the actions α_N and $\alpha_{G/N}$ are smooth (because every continuous homomorphism between Lie groups is analytic [He01, Ch. II, Theorem 2.6]). It should be noted that the f-invariant does not take on the value $+\infty$. So the previous lemma implies $f_{\mu_{G/N}}(\alpha_{G/N}) + f_{\mu_N}(\alpha_N) = -\infty$. □

4. Relative entropy

The proof of Theorem 3.3 uses the relative f-invariant theory developed in [Bo10c], which we review here. Let $(X, \mathcal{B}_X, \nu)$ be a standard probability space. Let $\mathcal{P}$ be a countable measurable partition of X and let $\mathcal{F} \subset \mathcal{B}_X$ be a sub-sigma algebra. Recall that for a.e. $x \in X$, the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}](x)$ is a probability measure on $(X, \mathcal{B}_X)$ satisfying

(1) $x \mapsto \mathbb{E}[A|\mathcal{F}](x)$ is $\mathcal{F}$-measurable for any $A \in \mathcal{B}_X$;
(2) $\int \mathbb{E}[A|\mathcal{F}](x)\ d\nu(x) = \nu(A)$ for any $A \in \mathcal{B}_X$.

The information function $I(\mathcal{P}|\mathcal{F})$ is a function on X defined by

$$I(\mathcal{P}|\mathcal{F})(x) = -\mathbb{E}[P_x|\mathcal{F}](x) \log(\mathbb{E}[P_x|\mathcal{F}](x))$$

where $P_x \in \mathcal{P}$ is the unique partition element with $x \in P_x$. The Shannon entropy of $\mathcal{P}$ *relative* to $\mathcal{F}$ is

$$H_\nu(\mathcal{P}|\mathcal{F}) = \int I(\mathcal{P}|\mathcal{F})(x)\ d\nu(x).$$

If T is a measure-preserving transformation of $(X, \mathcal{B}_X, \nu)$ then the entropy rate of $(T, \mathcal{P})$ *relative* to $\mathcal{F}$ is

$$h_\nu(T, \mathcal{P}|\mathcal{F}) = \lim_{n\to\infty} \frac{1}{2n+1} H_\nu\left(\bigvee_{i=-n}^{n} T^i\mathcal{P}|\ \mathcal{F}\right).$$

This is well-defined whenever $\mathcal{F}$ is T-invariant. We also define the entropy rate of T *relative* to $\mathcal{F}$ by

$$h_\nu(T|\mathcal{F}) = \sup_{\mathcal{P}} h_\nu(T, \mathcal{P}|\mathcal{F})$$

where the supremum is over all finite-entropy partitions $\mathcal{P}$ of X.

Now suppose $\Gamma = \langle s_1, \ldots, s_r\rangle$ and α is a measure-preserving action of Γ on $(X, \mathcal{B}_X, \nu)$. Define

$$\begin{aligned} F_\nu(\alpha, \mathcal{P}|\mathcal{F}) &= (1-2r)H_\nu(\mathcal{P}|\mathcal{F}) + \sum_{i=1}^{r} H_\nu(\mathcal{P} \vee \alpha_{s_i}\mathcal{P}|\mathcal{F}) \\ f_\nu(\alpha, \mathcal{P}|\mathcal{F}) &= \inf_{n>0} F_\nu\left(\alpha, \mathcal{P}^{B(n)}|\mathcal{F}\right). \end{aligned}$$

Also define

$$\begin{aligned} F^*_\nu(\alpha, \mathcal{P}|\mathcal{F}) &= (1-r)H_\nu(\mathcal{P}|\mathcal{F}) + \sum_{i=1}^{r} h_\nu(\alpha_{s_i}, \mathcal{P}|\mathcal{F}) \\ f^*_\nu(\alpha, \mathcal{P}|\mathcal{F}) &= \inf_{n>0} F^*_\nu\left(\alpha, \mathcal{P}^{B(n)}|\mathcal{F}\right). \end{aligned}$$

Theorem 4.1. *Let α be a measure-preserving action of Γ on a standard probability space $(X, \mathcal{B}_X, \nu)$. If $\mathcal{P}, \mathcal{Q}$ are any two finite-entropy generating partitions for α and $\mathcal{F} \subset \mathcal{B}_X$ is an $\alpha(\Gamma)$-invariant sub-σ-algebra then $f_\nu(\alpha, \mathcal{P}|\mathcal{F}) = f_\nu(\alpha, \mathcal{Q}|\mathcal{F}) = f^*_\nu(\alpha, \mathcal{P}|\mathcal{F}) = f^*_\nu(\alpha, \mathcal{Q}|\mathcal{F})$.*

Proof. This is implied by [Bo10c, Theorems 5.3, 9.1]. The proof requires a small correction; see [BG12]. □

Because of this theorem, we define $f_\nu(\alpha|\mathcal{F}) \triangleq f_\nu(\alpha,\mathcal{P}|\mathcal{F})$ where $\mathcal{P}$ is any finite-entropy generating partition for α. If there does not exist a finite-entropy generating partition for α then $f_\nu(\alpha|\mathcal{F})$ is undefined.

Theorem 4.2. *[The f-invariant Abramov-Rokhlin Addition Formula] Let α be a measure-preserving action of Γ on a standard probability space $(X,\mathcal{B}_X,\nu)$. Let $\mathcal{P},\mathcal{Q}$ be finite-entropy partitions of X. Let $\Sigma(\mathcal{Q})$ be the smallest Γ-invariant sub-σ-algebra containing $\mathcal{Q}$. Then*

$$f_\nu(\alpha,\mathcal{P}\vee\mathcal{Q}) = f_\nu(\alpha,\mathcal{Q}) + f_\nu(\alpha,\mathcal{P}|\Sigma(\mathcal{Q})).$$

Proof. This is [Bo10c, Theorem 1.3]. The proof requires a small correction; see [BG12]. □

5. A key Lemma

The purpose of this section is to prove the key lemma below for skew-products of $\mathbb{Z}$-actions. Let $(X,\mathcal{B}_X,\nu)$ be a Lebesgue space, $T\in \mathrm{Aut}(X,\mathcal{B}_X,\nu)$, G a compact metrizable group, equipped with Haar measure μ and S a group-automorphism of G. A cocycle for T and S is a cocycle for the actions of $\mathbb{Z}$ induced by T and S. That is, it is a measurable map $\sigma:\mathbb{Z}\times X\to \mathrm{G}$ such that

$$\sigma(n+m,x) = (S^n\sigma(m,x))\cdot\sigma(n,T^m x). \tag{5.1}$$

Lemma 5.1. *Let $(X,\mathcal{B}_X,\nu),\mathrm{G},T,S,\sigma$ be as above. Let $\mathcal{Q}$ be a finite-entropy partition of* G. *Let*

$$K(\mathcal{Q}) = \sup_{g\in \mathrm{G}} H(\mathcal{Q}g|\mathcal{Q}) + H(\mathcal{Q}|\mathcal{Q}g).$$

Then

$$\Big|h_{\nu\times\mu}(T\times_\sigma S, X\times\mathcal{Q}|\mathcal{B}_X) - h_\mu(S,\mathcal{Q})\Big| \le K(\mathcal{Q}).$$

Proof. By the definition of conditional entropy :

$$\begin{aligned}
& h_{\nu\times\mu}(T\times_\sigma S, X\times\mathcal{Q}|\mathcal{B}_X)\\
= \;& \lim_{m\to\infty}\frac{1}{m}H_{\nu\times\mu}\left(\bigvee_{k=0}^{m-1}(T\times_\sigma S)^{-k}X\times\mathcal{Q}|\mathcal{B}_X\right)\\
= \;& \lim_{m\to\infty}\frac{1}{m}\int H_{\delta_x\times\mu}\left(\bigvee_{k=0}^{m-1}(T\times_\sigma S)^{-k}X\times\mathcal{Q}\right)\,d\nu(x)
\end{aligned}$$

where δ_x is the Dirac probability measure concentrated on $\{x\}$.

We claim that for any set $P\subset \mathrm{G}$,

$$\{x\}\times\mathrm{G}\cap(T\times_\sigma S)^{-k}(X\times P) = \{x\}\times S^{-k}(P\sigma(k,x)^{-1}).$$

Indeed, (x,y) is contained in $(T\times_\sigma S)^{-k}(X\times P)$ if and only if

$$(T\times_\sigma S)^k(x,y) = (T^k x,(S^k y)\sigma(k,x))\in X\times P$$

which occurs if and only if

$$y\in S^{-k}(P\sigma(k,x)^{-1}).$$

So if

$$\mathcal{Q}_x^m = \bigvee_{k=0}^{m-1} S^{-k}(\mathcal{Q}\sigma(k,x)^{-1}).$$

then

$$H_{\delta_x \times \mu}\left(\bigvee_{k=0}^{m-1} (T \times_\sigma S)^{-k} X \times \mathcal{Q}\right) \quad = \quad H_\mu(\mathcal{Q}_x^m)$$

which implies:

$$h_{\nu\times\mu}((T \times_\sigma S), X \times \mathcal{Q}|\mathcal{B}_X) = \lim_{m\to\infty} \frac{1}{m} \int_X H_\mu(\mathcal{Q}_x^m) d\nu(x) \tag{5.2}$$

Define:

$$\mathcal{Q}^m = \bigvee_{k=0}^{m-1} S^{-k}\mathcal{Q}$$

By the definition of entropy:

$$h_\mu(S,\mathcal{Q}) = \lim_{m\to\infty} \frac{1}{m} \int_X H_\mu(\mathcal{Q}^m) d\nu(x) \tag{5.3}$$

Note $|H_\mu(\mathcal{Q}^m) - H_\mu(\mathcal{Q}_x^m)| \le H_\mu(\mathcal{Q}^m|\mathcal{Q}_x^m) + H_\mu(\mathcal{Q}_x^m|\mathcal{Q}^m)$. Thus:

$$\begin{aligned} |H_\mu(\mathcal{Q}^m) - H_\mu(\mathcal{Q}_x^m)| &\le \sum_{k=0}^{m-1} H_\mu(S^{-k}\mathcal{Q}|S^{-k}(\mathcal{Q}\sigma(k,x)^{-1})) + H_\mu(S^{-k}(\mathcal{Q}\sigma(k,x)^{-1})|S^{-k}\mathcal{Q}) \\ &= \sum_{k=0}^{m-1} H_\mu(\mathcal{Q}|\mathcal{Q}\sigma(k,x)^{-1}) + H_\mu(\mathcal{Q}\sigma(k,x)^{-1})|\mathcal{Q}) \le mK(\mathcal{Q}). \end{aligned}$$

Finally (5.2) and (5.3) imply $|h_{\nu\times\mu}((T \times_\sigma S), X \times \mathcal{Q}|\mathcal{B}_X) - h_\mu(S,\mathcal{Q})| \le K(\mathcal{Q})$. □

6. Proof of Theorem 3.3

For this section, let $\Gamma, (X, \mathcal{B}_X, \nu), (\mathrm{G}, \mathcal{B}_G, \mu), \alpha, \beta, \sigma$ be as in Theorem 3.3. A *special partition* of G is a partition $\mathcal{Q}$ such that there exists a finite-index normal clopen subgroup $N < \mathrm{G}$ such that $\mathcal{Q} = \{gN : g \in \mathrm{G}\}$. The next lemma is left as an exercise for the reader.

Lemma 6.1. *If $\mathcal{Q}$ is special and $T_1, \ldots, T_n$ are automorphisms of* G *then $\bigvee_{i=1}^n T_i\mathcal{Q}$ is also special.*

Lemma 6.2. *If $\mathcal{P}$ is any finite-entropy partition of X and $\mathcal{Q}$ is a special partition of G then*

$$F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, \mathcal{P} \times \mathcal{Q}|\mathcal{B}_X) = F^*_\mu(\beta, \mathcal{Q}).$$

Proof. Because $\mathcal{Q}g = \mathcal{Q}$ for any $g \in G$, it follows that $K(\mathcal{Q}) = 0$ where $K(\cdot)$ is as defined in Lemma 5.1. So that Lemma implies

$$\begin{aligned} F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, \mathcal{P} \times \mathcal{Q}|\mathcal{B}_X) &= (1-r)H_{\nu\times\mu}(\mathcal{P} \times \mathcal{Q}|\mathcal{B}_X) + \sum_{i=1}^r h_{\nu\times\mu}((\alpha \times_\sigma \beta)_{s_i}, \mathcal{P} \times \mathcal{Q}|\mathcal{B}_X) \\ &= (1-r)H_\mu(\mathcal{Q}) + \sum_{i=1}^r h_\mu(\beta_{s_i}, \mathcal{Q}) = F^*_\mu(\beta, \mathcal{Q}). \end{aligned}$$

□

Lemma 6.3. *Let $\mathcal{Q}$ be a special partition of G, $g \in \Gamma$ and $\mathcal{P}_g$ denote the partition of X obtained by pulling $\beta_g(\mathcal{Q})$ back under the cocycle $\sigma(g, \cdot)$. Also, let $\mathcal{P}'$ be an arbitrary measurable partition of X. Then*

$$(\alpha \times_\sigma \beta)_g((\mathcal{P}_g \vee \mathcal{P}') \times \mathcal{Q}) = \alpha_g(\mathcal{P}_g \vee \mathcal{P}') \times \beta_g(\mathcal{Q})$$

(up to sets of measure zero).

Proof. Let N be the finite-index clopen normal subgroup of G such that $\mathcal{Q} = \{qN :\ q \in G\}$. Let $P \in \mathcal{P}_g, P' \in \mathcal{P}'$ and $qN \in \mathcal{Q}$. It suffices to show that there exists some $q'' \in G$ such that

$$(\alpha \times_\sigma \beta)_g((P \cap P') \times qN) = \alpha_g(P \cap P') \times q''\beta_g(N)$$

up to sets of measure zero. By definition of $\mathcal{P}_g$, there exists a coset $q'\beta_g(N) \in G/\beta_g(N)$ such that for every $y \in P$, $\sigma(g, y) \in q'\beta_g(N)$.

Let $x \in P \cap P'$ and $n \in N$. Then there exists some $m \in N$ such that

$$(\alpha \times_\sigma \beta)_g(x, qn) = (\alpha_g x, \beta_g(qn)\sigma(g, x)) = (\alpha_g x, \beta_g(qn)q'\beta_g(m)).$$

Because N is normal, $\beta_g(qn)q'\beta_g(m) \in \beta_g(q)q'\beta_g(N)$. Thus $(\alpha \times_\sigma \beta)_g(x, qn) \in \alpha_g(P \cap P') \times \beta_g(q)q'\beta_g(N)$. Since $(\alpha \times_\sigma \beta)_g$ preserves $\nu \times \mu$, it follows that $(\alpha \times_\sigma \beta)_g((P \cap P') \times qN) = \alpha_g(P \cap P') \times q''\beta_g(N)$ up to sets of measure zero. □

Lemma 6.4. *Let $\mathcal{P}, \mathcal{Q}$ be measurable partitions for α, β respectively. Suppose $\mathcal{Q}$ is special and $\mathcal{P}$ is generating. Let $\Sigma(\mathcal{P}, \mathcal{Q})$ be the smallest $\alpha \times_\sigma \beta(\Gamma)$-invariant σ-algebra containing $\mathcal{P} \times \mathcal{Q}$. Similarly, let $\Sigma(\mathcal{Q})$ be the smallest $\beta(\Gamma)$-invariant σ-subalgebra of $\mathcal{B}_G$ which contains $\mathcal{Q}$.*

Then $\Sigma(\mathcal{P}, \mathcal{Q})$ is the smallest σ-algebra containing $\mathcal{B}_X \times \Sigma(\mathcal{Q})$ (up to sets of measure zero).

Proof. Clearly, $\mathcal{P} \times G$ is contained in $\Sigma(\mathcal{P}, \mathcal{Q})$. Because

$$(\alpha \times_\sigma \beta)_g(\mathcal{P} \times \mathrm{G}) = (\alpha_g \mathcal{P}) \times G, \quad \forall g \in \Gamma,$$

it follows that $(\alpha_g \mathcal{P}) \times G \subset \Sigma(\mathcal{P}, \mathcal{Q})$ for every $g \in \Gamma$. Because $\mathcal{P}$ is generating, this implies $\mathcal{B}_X \times G \subset \Sigma(\mathcal{P}, \mathcal{Q})$ (up to sets of measure zero).

For each $g \in \Gamma$, recall that $\mathcal{P}_g$ is the partition of X obtained by pulling $\beta_g(\mathcal{Q})$ back under the cocycle $\sigma(g, \cdot)$. Because $\sigma(g, \cdot)$ is $\mathcal{B}_X$-measurable, $\mathcal{P}_g \times \mathcal{Q}$ is contained in $\Sigma(\mathcal{P}, \mathcal{Q})$. By Lemma 6.3,

$$(\alpha \times_\sigma \beta)_g(\mathcal{P}_g \times \mathcal{Q}) = (\alpha_g \mathcal{P}_g) \times (\beta_g \mathcal{Q}) \subset \Sigma(\mathcal{P}, \mathcal{Q})$$

(up to sets of measure zero). Because $X \times \beta_g \mathcal{Q}$ coarsens $(\alpha_g \mathcal{P}_g) \times (\beta_g \mathcal{Q})$, it follows that $X \times \beta_g \mathcal{Q} \subset \Sigma(\mathcal{P}, \mathcal{Q})$ for every $g \in \Gamma$. By definition of $\Sigma(\mathcal{Q})$, this implies $X \times \Sigma(\mathcal{Q}) \subset \Sigma(\mathcal{P}, \mathcal{Q})$.

Because $X \times \Sigma(\mathcal{Q})$ and $\mathcal{B}_X \times G$ generate $\mathcal{B}_X \times \Sigma(\mathcal{Q})$ (up to sets of measure zero), this implies $\Sigma(\mathcal{P}, \mathcal{Q}) \supset \mathcal{B}_X \times \Sigma(\mathcal{Q})$.

To show the opposite inclusion, it suffices to show that $(\alpha \times_\sigma \beta_g)(\mathcal{P} \times \mathcal{Q}) \in \mathcal{B}_X \times \Sigma(\mathcal{Q})$ for any $g \in \Gamma$. By the previous lemma,

$$(\alpha \times_\sigma \beta_g)(\mathcal{P} \times \mathcal{Q}) \le (\alpha \times_\sigma \beta)_g((\mathcal{P}_g \vee \mathcal{P}) \times \mathcal{Q}) = (\alpha_g(\mathcal{P}_g \vee \mathcal{P})) \times (\beta_g \mathcal{Q}) \in \mathcal{B}_X \times \Sigma(\mathcal{Q}).$$

□

Proof of Theorem 3.3. Let $\mathcal{P}$ be a finite-entropy generating partition for α and $\mathcal{Q}$ be a special generating partition for β. By the previous lemma, $\mathcal{P} \times \mathcal{Q}$ is generating for $\alpha \times_\sigma \beta$. So Theorems 4.1 and 4.2 imply

$$\begin{aligned} f_{\nu\times\mu}(\alpha \times_\sigma \beta) - f_\nu(\alpha) &= f_{\nu\times\mu}(\alpha \times_\sigma \beta | \mathcal{B}_X) \\ &= \inf_{n>0} F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, (\mathcal{P} \times \mathcal{Q})^{B(n)} | \mathcal{B}_X). \end{aligned}$$

For each $g \in \Gamma$, let $\mathcal{P}_g$ be the partition of X obtained by pulling $(\beta_g \mathcal{Q})$ back under $\sigma(g, \cdot)$. By Lemma 6.3, for any partition $\mathcal{P}'$ of X

$$(\alpha \times_\sigma \beta)_g((\mathcal{P}_g \vee \mathcal{P}') \times \mathcal{Q}) = (\alpha_g(\mathcal{P}_g \vee \mathcal{P}')) \times (\beta_g \mathcal{Q}).$$

Given an integer $n > 0$ let $\mathcal{R}_n = \bigvee_{g \in B(n)} \mathcal{P}_g$. By Lemma 6.3,

$$\begin{aligned} (\mathcal{P} \vee \mathcal{R}_n) \times \mathcal{Q})^{B(n)} &= \bigvee_{w \in B(n)} (\alpha \times_\sigma \beta)_w((\mathcal{P} \vee \mathcal{R}_n) \times \mathcal{Q}) \\ &= \bigvee_{w \in B(n)} \alpha_w(\mathcal{P} \vee \mathcal{R}_n) \times \beta_w(\mathcal{Q}) \\ &= (\mathcal{P} \vee \mathcal{R}_n)^{B(n)} \times \mathcal{Q}^{B(n)}. \end{aligned}$$

Because we are conditioning on $\mathcal{B}_X$ and $(\mathcal{R}_n \times G)^{B(n)} = (\mathcal{R}_n^{B(n)} \times G)$,

$$\begin{aligned} F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, (\mathcal{P} \times \mathcal{Q})^{B(n)} | \mathcal{B}_X) &= F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, (\mathcal{P} \times \mathcal{Q})^{B(n)} \vee \mathcal{R}_n^{B(n)} \times G | \mathcal{B}_X) \\ &= F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, ((\mathcal{P} \vee \mathcal{R}_n) \times \mathcal{Q})^{B(n)} | \mathcal{B}_X) \\ &= F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, ((\mathcal{P} \vee \mathcal{R}_n)^{B(n)} \times \mathcal{Q}^{B(n)} | \mathcal{B}_X). \end{aligned}$$

By Lemma 6.2,

$$F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, ((\mathcal{P} \vee \mathcal{R}_n)^{B(n)} \times \mathcal{Q}^{B(n)} | \mathcal{B}_X) = F^*_\mu(\beta, \mathcal{Q}^{B(n)}).$$

So we now have

$$\begin{aligned} f_{\nu\times\mu}(\alpha \times_\sigma \beta) &= f_\nu(\alpha) + f_{\nu\times\mu}(\alpha \times_\sigma \beta | \mathcal{B}_X) \\ &= f_\nu(\alpha) + \inf_{n>0} F^*_{\nu\times\mu}(\alpha \times_\sigma \beta, (\mathcal{P} \times \mathcal{Q})^{B(n)} | \mathcal{B}_X) \\ &= f_\nu(\alpha) + \inf_{n>0} F^*_\mu(\beta, \mathcal{Q}^{B(n)}) \\ &= f_\nu(\alpha) + f_\mu(\beta). \end{aligned}$$

The last equality holds by Theorem 2.1. □

7. Examples

It is convenient to introduce the following notation. Let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be the rank r free group. If K is a set then K^Γ is the set of all functions $x : \Gamma \to K$. The *shift-action* of Γ on K^Γ is defined as follows. For $g, f \in \Gamma$ and $x \in K^\Gamma$, $gx \in K^\Gamma$ is the map $(gx)(f) = x(g^{-1}f)$.

If Γ acts on a compact group G and the action is understood, we write $f(\Gamma \curvearrowright G)$ to mean the f-invariant of the action of G with respect to Haar measure.

7.1. The Ornstein-Weiss Example.

This example comes from the appendix to [OW87]. To explain its relevance, let us recall some basic facts from classical entropy theory. Let Δ be a countable amenable group, K a finite set and u the uniform probability measure on K. It is straightforward to compute the entropy of the shift action of Δ on (K^Δ, u^Δ): it is $\log |K|$. Because entropy never increases under a factor map, it follows that if $|K| > 1$ then the action $\Delta \curvearrowright (K^\Delta, u^\Delta)$ cannot factor onto the action $\Delta \curvearrowright ((K \times K)^\Delta, (u \times u)^\Delta)$.

By contrast, Ornstein and Weiss showed that if Γ is the rank 2 free group then $\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z})^\Gamma$ factors onto $\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma$. This convinced many researchers that there could not be an entropy theory for free groups.

The factor map is defined by

$$\phi : (\mathbb{Z}/2\mathbb{Z})^\Gamma \to (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma,$$

$$\phi(x)(g) = (x(g) + x(gs_1), x(g) + x(gs_2)), \forall x \in (\mathbb{Z}/2\mathbb{Z})^\Gamma, g \in \Gamma.$$

We consider $(\mathbb{Z}/2\mathbb{Z})^\Gamma$ and $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma$ as compact groups under pointwise addition. It is a straightforward exercise to show that ϕ is a surjective homomorphism which is equivariant with respect to the shift-actions of Γ and therefore, defines a factor map. Moreover, the kernel of ϕ consists of two elements, x_0, x_1, where $x_i : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ is defined by $x_i(g) = i$. Let $N = \{x_0, x_1\}$. Because N is finite, it clearly satisfies the conditions of Theorem 2.2. So that result implies

$$f(\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z})^\Gamma) = f(\Gamma \curvearrowright N) + f(\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma).$$

In [Bo10a], it is shown that $f(\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z})^\Gamma) = \log(2)$ and $f(\Gamma \curvearrowright (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^\Gamma) = \log(4)$ as expected. Therefore, $f(\Gamma \curvearrowright N) = -\log(2)$. This is easy to verify by direct computation.

7.2. A generalization.

The example above can be generalized with the help of [MRV11, proof of Theorem B] which states the following: if $\Gamma = \langle s_1, \ldots, s_r \rangle$ is any finite rank free group, K is any compact second countable group, K^Γ is the group of all functions $x : \Gamma \to K$ under pointwise multiplication and K is identified with the constant functions in K^Γ then the action $\Gamma \curvearrowright K^\Gamma / K$ is measurably conjugate to $\Gamma \curvearrowright (K^r)^\Gamma$ (where the measures involved are the Haar measures and the actions are the shift actions).

When K is finite, we can apply Theorem 2.2 to obtain

$$f(\Gamma \curvearrowright K^\Gamma) = f(\Gamma \curvearrowright K) + f(\Gamma \curvearrowright (K^r)^\Gamma).$$

This is easy to check: $f(\Gamma \curvearrowright K^\Gamma) = \log(|K|)$ and $f(\Gamma \curvearrowright (K^r)^\Gamma) = r \log(|K|)$ by [Bo10a]. By a straightforward computation, $f(\Gamma \curvearrowright K) = -(r-1)\log |K|$.

7.3. **An algebraic example.** As above, let $\Gamma = \langle s_1, \ldots, s_r \rangle$ be a finite rank free group. Let $p > 1$ be a prime number and $h \in (\mathbb{Z}/p\mathbb{Z})\Gamma$. We consider h as a function from Γ to $\mathbb{Z}$ such that $h(s) = 0$ for all but finitely many $s \in \Gamma$. Define the convolution operator $\phi_h : (\mathbb{Z}/p\mathbb{Z})^\Gamma \to (\mathbb{Z}/p\mathbb{Z})^\Gamma$ by

$$\phi_h(x)(g) = \sum_{s \in \Gamma} x(gs)h(s^{-1}), \quad \forall g \in \Gamma.$$

This is a Γ-equivariant homomorphism. Let $X_{h,p}$ denote the kernel of ϕ_h. Let $X^*_{h,p} < X_{h,p}$ be the subgroup consisting of all elements $x \in X_{h,p}$ with $x(e) = 0$. This is a finite-index normal clopen subgroup and $\{gX^*_{h,p} : \ g \in X_{h,p}\}$ is a generating partition for the shift-action of Γ. Therefore, we can apply Theorem 2.2 to obtain

$$f(\Gamma \curvearrowright (\mathbb{Z}/p\mathbb{Z})^\Gamma) = f(\Gamma \curvearrowright X_{h,p}) + f(\Gamma \curvearrowright \phi_h((\mathbb{Z}/p\mathbb{Z})^\Gamma)).$$

Theorem 7.1. *ϕ_h is onto if h is nonzero.*

Therefore,

$$f(\Gamma \curvearrowright \phi_h((\mathbb{Z}/p\mathbb{Z})^\Gamma)) = f(\Gamma \curvearrowright (\mathbb{Z}/p\mathbb{Z})^\Gamma).$$

Thus $f(\Gamma \curvearrowright X_{h,p}) = 0$.

To prove Theorem 7.1, we need a little preparation.

Definition 7.2. Let C_r be the Cayley graph of Γ. It has vertex set Γ and edges $\{g, gs_i\}$ for all $g \in \Gamma$ and $1 \leq i \leq r$. Given a set $F \subset \Gamma$, the *induced subgraph* of F is the subgraph $C_r(F) \subset C_r$ which has vertex set F and contains every edge of C_r which has both endpoints in F. A subset $F \subset \Gamma$ is said to be *connected* if its induced subgraph in C_r is connected. The *convex hull* of a set $F \subset \Gamma$ is the smallest connected set $F' \subset \Gamma$ with $F \subset F'$. An *extreme point* of F is an element $f \in F$ that has degree 1 in $C_r(F)$. We let $\mathrm{Ex}(F)$ denote the set of extreme points of F. Note that if F' is the convex hull of F then $\mathrm{Ex}(F') \subset F$.

Lemma 7.3. *Let $F = \{g \in \Gamma : \ h(g^{-1}) \neq p\mathbb{Z}\}$. Let $\overline{F}$ be the convex hull of F. Suppose there exists an ordering $\gamma_0, \gamma_1, \gamma_2, \ldots$ of Γ such that for every $n \geq 1$ $\{\gamma_0, \ldots, \gamma_n\}$ is connected and*

$$\gamma_n \overline{F} \nsubseteq \cup_{i=0}^{n-1} \gamma_i \overline{F}.$$

Then ϕ_h is onto.

Proof. By compactness of $(\mathbb{Z}/p\mathbb{Z})^\Gamma$ and continuity of ϕ_h, it suffices to show that for every $y \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ and every $n \geq 0$, there exists an $x \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ such that $\phi_h(x)(\gamma_i) = y(\gamma_i)$ for every $0 \leq i \leq n$. We will prove this statement by induction on n. It is clearly true for $n = 0$. So suppose there is an $n \geq 0$ for which the statement is true. Fix $y \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ and let $x \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ be such that $\phi_h(x)(\gamma_i) = y(\gamma_i)$ for every $0 \leq i \leq n$.

By hypothesis, $\gamma_{n+1}\overline{F} \nsubseteq \cup_{i=0}^{n} \gamma_n \overline{F}$. Because $\cup_{i=0}^{n} \gamma_n \overline{F}$ and $\gamma_{n+1}\overline{F}$ are connected and the convex hull of the exreme points set of a connected set is the connected set itself , there must be an extremal point $f \in \mathrm{Ex}(\overline{F})$ such that $\gamma_{n+1} f \notin \cup_{i=0}^{n} \gamma_n \overline{F}$. However, $\mathrm{Ex}(\overline{F}) \subset F$. So $f \in F$. By definition, this means that $h(f^{-1}) \neq p\mathbb{Z}$. Because p is prime, we may therefore define an element $m \in \mathbb{Z}/p\mathbb{Z}$ by

$$m = h(f^{-1})^{-1} \left(y(\gamma_{n+1}) - \sum_{g \in \Gamma \setminus \{f\}} x(\gamma_{n+1} g) h(g^{-1}) \right).$$

Define $x' \in (\mathbb{Z}/p\mathbb{Z})^\Gamma$ by $x'(g) = x(g)$ if $g \neq \gamma_{n+1} f$ and $x'(\gamma_{n+1} f) = m$. Because $\gamma_{n+1} f \notin \cup_{i=0}^n \gamma_n \overline{F}$, it follows that $\phi_h(x')(\gamma_i) = \phi_h(x)(\gamma_i)$ for all $0 \le i \le n$. Also a straightforward computation shows $\phi_h(x')(\gamma_{n+1}) = y(\gamma_{n+1})$. So $\phi_h(x')(\gamma_i) = y(\gamma_i)$ for all $0 \le i \le n+1$. This completes the inductive step and the claim. $\square$

Definition 7.4. Let $S = \{s_1, \ldots, s_r\}$. For $g \in \Gamma$, let $|g|$ be the smallest number $n \ge 0$ such that there exist elements $t_1, \ldots, t_n \in S \cup S^{-1}$ with $g = t_1 \cdots t_n$. We also let $d(g_1, g_2) = |g_1^{-1} g_2|$ for any $g_1, g_2 \in \Gamma$. For $g \in \Gamma$ and $n \ge 0$, let $B(g,n) = \{k \in \Gamma : \ d(k,g) \le n\}$ be the ball of radius n centered at g.

Let $K \subset \Gamma$ be a finite set. The *radius* of K is the smallest number $r \ge 0$ such that there exists a $v \in \Gamma$ such that $B(v,r) \supset K$. An element $v \in \Gamma$ is called a *center* of K if $B(v,r) \supset K$ where r is the radius of K. For any $v, w \in \Gamma$, we let $[v,w] \subset \Gamma$ be the set of all $g \in \Gamma$ such that the shortest path from v to w in the Cayley graph C_r contains g.

Lemma 7.5. *Let K be a connected finite set with radius $r \ge 1$. Suppose the identity element e is a center of K. Then there exist elements $v, w \in K$ such that $[e,v] \cap [e,w] = \{e\}$, $|v| = r$ and $|w| \in \{r-1, r\}$.*

Proof. Because K has radius r and center e, there is an element v with $|v| = r$. To obtain a contradiction, suppose that there is no $w \in K$ with $|w| \in \{r-1, r\}$ and $[e,v] \cap [e,w] = \{e\}$. Let $v_1 \in S \cup S^{-1}$ be the unique element with $|v_1^{-1} v| = r-1$. We claim that $B(v_1, r-1) \supset K$. To see this, let $w \in K$. If $|w| \le r-2$ then $w \in B(e, r-2) \subset B(v_1, r-1)$. If $|w| > r-2$ then, because K has center e and radius r, $|w| \in \{r-1, r\}$. By assumption, this implies $[e,v] \cap [e,w] \neq \{e\}$. So let $y \in [e,v] \cap [e,w]$ with $y \neq e$. Then $[e,y] \subset [e,v]$. This implies that $v_1 \in [e,y]$. In particular, $v_1 \in [e,v] \cap [e,w]$, so $v_1 \in [e,w]$. Because $|w| \le r$, this implies $d(v_1, w) \le r-1$ as claimed. So we have shown that in all cases, if $w \in K$ then $w \in B(v_1, r-1)$. This shows that the radius of K is at most $r-1$, a contradiction. This contradiction proves the lemma. $\square$

Lemma 7.6. *Let K be a connected finite set with radius $r \ge 1$. Suppose the identity element e is a center of K. Suppose $g_1, \ldots, g_n \in \Gamma \setminus \{e\}$ are elements with*

$$K \subset \cup_{i=1}^n g_n K.$$

Then e is contained in the convex hull of $\{g_1, \ldots, g_n\}$.

Proof. Let $v, w \in K$ be elements such that $[e,v] \cap [e,w] = \{e\}$, $|v| = r$ and $|w| \in \{r-1, r\}$. Let $g_i, g_j \in \{g_1, \ldots, g_n\}$ be such that $v \in g_i K$ and $w \in g_j K$. Let $x, y \in K$ be such that $v = g_i x$ and $w = g_j y$.

Let $v_1, v_2, x_1, x_2 \in \Gamma$ be such that $v = v_1 v_2$, $|v| = |v_1| + |v_2|$, $x_2 = v_2$, $x = x_1 x_2$, $|x| = |x_1| + |x_2|$ and $|v_2| = |x_2|$ is as large as possible. Thus $g_i = v x^{-1} = v_1 x_1^{-1}$ and $|v x^{-1}| = |v_1| + |x_1|$. Because r is the radius of K, e is a center and $x \in K$ we have $|x| \le r$. Also, we cannot have $v = x$ (since this would imply $g_i = v x^{-1} = e$, a contradiction). So we must have $|v_1| \ge 1$. Thus $[e,v] \cap [e,g_i] \neq \{e\}$.

Let $w_1, w_2, y_1, y_2 \in \Gamma$ be such that $w = w_1 w_2$, $|w| = |w_1| + |w_2|$, $y_2 = w_2$, $y = y_1 y_2$, $|y| = |y_1| + |y_2|$ and $|w_2| = |y_2|$ is as large as possible. Thus $g_j = w y^{-1} = w_1 y_1^{-1}$ and $|w y^{-1}| = |w_1| + |y_1|$. Because r is the radius of K, e is a center and $y \in K$ we have $|y| \le r$.

Case 1. If $|w| = r$, then, as in the previous paragraph, we must have $[e,w] \cap [e,g_j] \neq \{e\}$. Because $[e,v] \cap [e,w] = \{e\}$, this implies $e \in [g_i, g_j]$ which implies the lemma.

Case 2. Suppose $|w| = r-1$ and $|w_1| \geq 1$. Thus $[e,w] \cap [e,g_j] \neq \{e\}$. Because $[e,v] \cap [e,w] = \{e\}$, this implies $e \in [g_i, g_j]$ which implies the lemma.

Case 3. Suppose $|w| = r-1$ and $|w_1| = 0$. Then $w = w_2$, so $|w_2| = r-1$. Because $g_j = wy^{-1} = w_1 y_1^{-1} = y_1^{-1} \neq e$, we must $y_1 \neq e$. Thus $|y| = |y_1| + |y_2| = |y_1| + |w_2| = |y_1| + r - 1$. Because $y \in K$ and K has radius r and center e, we must have $|y_1| = 1$ and $|y| = r$. If $[e,y] \cap [e,v] = \{e\}$ then, after replacing w with y we are in the situation of Case 1 (note $y = g_k y'$ for some $1 \leq k \leq n$ and $y' \in K$). So we may assume $[e,y] \cap [e,v] \neq \{e\}$ which implies $y_1 \in [e,v]$. Because $g_j = y_1^{-1}$, and $[e,v] \cap [e,g_i] \neq \{e\}$, we have $[e,g_i] \cap [e,g_j] = \{e\}$ which implies $e \in [g_i,g_j]$ which implies the lemma.

[Proof of Theorem 7.1] Let $F = \{g \in \Gamma : \ h(g^{-1}) \neq p\mathbb{Z}\}$. Let $\overline{F}$ be the convex hull of F. For any $g \in \Gamma$, ϕ_h is onto if and only if ϕ_{gh} is onto. So after replacing h with gh for some $g \in \Gamma$, we may assume that e is a center of $\overline{F}$.

Let $g_0, g_1, \ldots$ be an ordering of Γ such that for every $n \geq 0$, $\{g_0, \ldots, g_n\}$ is connected. We claim that for every $n \geq 1$,

$$\gamma_n \overline{F} \nsubseteq \cup_{i=0}^{n-1} \gamma_i \overline{F}.$$

To obtain a contradiction, suppose that the claim is false for some $n \geq 1$. Then $\overline{F} \subset \cup_{i=0}^{n-1} \gamma_n^{-1}\gamma_i \overline{F}$, $\gamma_n^{-1}\gamma_i \neq e$ for any $0 \leq i \leq n-1$ and because $\{\gamma_0, \ldots, \gamma_{n-1}\}$ is connected, $\{\gamma_n^{-1}\gamma_0, \ldots, \gamma_n^{-1}\gamma_{n-1}\}$ is connected which implies that e is not in the convex hull of $\{\gamma_n^{-1}\gamma_0, \ldots, \gamma_n^{-1}\gamma_{n-1}\}$. This contradicts the previous lemma. So we must have that for every $n \geq 1$,

$$\gamma_n \overline{F} \nsubseteq \cup_{i=0}^{n-1} \gamma_i \overline{F}.$$

The theorem now follows from Lemma 7.3. □

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